Abstract

Optimal stopping, or selecting the optimal time to perform an action, is crucial when trading financial securities. Prophet inequalities, a subproblem in optimal stopping, compare the profits of online strategies to the profit of an offline strategy that knows the future and always acts optimally. It is natural to consider whether results related to prophet inequalities can be usefully applied in the realm of trading.

To keep this paper tractable, we model price movements of these securities as Brownian motions with negative drift. While stopping rules are trivial for standard financial securities (i.e. stocks and futures) under these circumstances, it is less trivial to identify an optimal stopping rule for option exercise. I show how to compute an optimal stopping rule for option exercise, develop and implement an algorithm following this rule, and observe empirical competitive ratios obtained by this strategy. Notably, this algorithm can be extended to use arbitrary probability distributions for underlying price movements, allowing us to compute expected values even in the case where prices do not follow Brownian motions.

1 Introduction

When it comes to investing, we all know the age-old maxim “buy low, sell high.” Empirically, however, this is much harder to do. For example, poorly timed trades, can lead to undesired “buy high, sell low” decisions that lose money for the investor. Even profitable trades can lead to regret in hindsight; a flighty investor may exit their position early, pocketing a few dollars, then wish they held longer as the security continued to grow. Of course, had the investor held their position, it may also have been possible for the security to fall, leaving the investor wishing they had exited their position earlier for a profit. Problems like this are fundamental in trading and lead to the question: “How high is high enough?” That is, when should a trader exit their position to maximize their expected profits?

Optimal stopping theory largely studies this class of problems – namely, when to take action, or stop, to optimize a certain value – and has application in a wide range of domains including the problem described above. We can abstract the problem as follows. Given a sequence of closed boxes, we open the boxes sequentially to reveal an unknown amount of money. Upon opening a box, we can either claim the money within and leave or reject the current box and open the next. However, once a box has been rejected, we cannot return to the box to reclaim the money. The parallels between this abstraction and the problem of knowing when to exit a position are clear; in both problems, we are offered a known, current reward and must choose to accept (and end the sequence) or reject the money, in favor of unknown, future rewards.
The example above is rather informal, but we provide a more formal problem below that falls within the general realm of prophet inequalities. Let $X_1, \ldots, X_n$ be a finite sequence of nonnegative independent random variables. A gambler, knowing the distributions of each $X_i$ and the number of variables $n$, samples the variables in order. After sampling $X_1$, the gambler can choose either to accept $X_1$ and end the game, or to reject $X_1$ and continue to $X_2$. Similarly, after sampling $X_2$ (assuming the gambler rejects $X_1$), the gambler must either accept or reject the observed value. We continue in this manner; should the gambler reject all variables $X_1, \ldots, X_{i-1}$, they observe $X_i$ and can accept or reject the value. Upon reaching $X_n$, the gambler must accept. Additionally, suppose there exists a prophet that knows the realizations (actual values) of each $X_i$ beforehand and trivially selects the maximum realization. Then, given a strategy for the gambler, we are interested in knowing how the winnings of the gambler compare to that of the prophet. For a given sequence of random variables, we say that the strategy has a competitive ratio of $\alpha$ if the gambler can expect to win $\alpha$ that of the prophet. We call a variant of the prophet inequality a set of rules or conditions on the sequence of random variables (e.g. all variables are nonnegative and i.i.d.). Then, given a variant, we can define the pathological competitive ratio of a strategy to be the infimum of the competitive ratios attained by the strategy for all possible sequences of random variables that are valid under the variant. If a strategy has a pathological competitive ratio of $\alpha$, we say it is $\alpha$-competitive. Naturally, we are interested in bounding the pathological competitive ratio above; this can be done by identifying some sequence of variables such that no strategy can achieve a competitive ratio greater than some $\alpha$. If there exists a strategy with a pathological competitive ratio equal to this upper bound, we say that the upper bound is tight.

These types of solutions (i.e. identifying pathological competitive ratios and upper bounds for specific variants) are known as prophet inequalities. Krengel and Sucheston [10, 11] studied the original prophet inequality, which required only that the sequence of random variables was finite, nonnegative, and independent. In particular, [10] identified a $1/2$-competitive strategy. That is letting $X_r$ and $\bar{X} = \mathbb{E}[\max_i]$ be the winnings of the gambler and the prophet, respectively, we know that

$$\mathbb{E}[X_r] \geq \frac{1}{2} \cdot \mathbb{E}[\bar{X}].$$

While the proof establishing the pathological competitive ratio of $1/2$ is fairly involved, the strategy itself is quite simple: accept the first sample that exceeds $\mathbb{E}[\bar{X}]/2$. Notably, $1/2$ is a tight upper bound on the original prophet inequality; in the worst case, the gambler cannot expect to win more than half of what the prophet wins. To show this, we provide a simple example that bounds the above strategy to a competitive ratio of $1/2$. Consider two random variables $X_1$ and $X_2$ where $X_1 = 1$ while

$$X_2 = \begin{cases} 
1/p & \text{w.p. } p \in (0, 1], \\
0 & \text{w.p. } 1 - p.
\end{cases}$$

The gambler can only expect to make 1 in either case, thus with any strategy $\mathbb{E}[X_r] = 1$. However, the prophet expects $\mathbb{E}[\bar{X}] = p(1/p) + (1 - p) = 2 - p$. Then $\mathbb{E}[\bar{X}] \rightarrow 2$ as $p \rightarrow 0$, meaning that the competitive ratio approaches $1/2$ as well.

Many techniques are used to create trading systems, in part due to the wide selection of financial assets available to investors; in the average brokerage, investors commonly have
access to stocks, bonds, futures, and options, not to mention the vast array of securities that fall within each category. As realistic market conditions are messy and difficult to model, many standard financial models (reviewed in Section 2.1) treat price movements as random walks. While many standard financial models such as Black-Scholes [2] use geometric Brownian motions to model price movements, we use (non-geometric) Brownian motions to model price movements to keep this paper tractable. However, note that stopping rules for stocks, bonds, and futures are trivial in this context. If drift (informally, the expected movement of the process) is positive, we simply hold indefinitely as that yields the greatest expected value, and if drift is zero or less, we simply sell immediately to avoid expected losses. Options, a financial product that grant the owner the right to trade a security at a specified price and time, are unique in that payoffs are strictly nonnegative. For example, consider an option that granted its owner the right to purchase a stock for $100 at any time. If the underlying stock rises to $150, the owner of the option may exercise the option, rendering it invalid, and purchase the stock for $100, then immediately sell it at the market price of $150, pocketing $50. On the other hand, if the underlying stock falls to $50, the owner of the option can simply let the option expire, also rendering the option invalid, suffering no loss. While stopping rules for option exercise are trivial in the case that drift is nonnegative (simply hold the option until contract expiry to maximize expected value), stopping rules for negative drift are less trivial as zero-valued options may be profitable in the future and as immediate exercise of positively valued options may not always be optimal. Inspired by previous work on prophet inequalities, we compute an optimal stopping rule for option exercise, develop an algorithm that implements this stopping rule, and identify empirical competitive ratios for the stopping rule under varying lengths, drifts, and variances of the underlying Brownian motion.

The remainder of this paper is organized as follows. In Section 2, we review basic options theory, provide theorems from probability theory we use in developing our algorithm, and survey common variants of the prophet inequality. In Section 3, we address the limitations of applying these variants to trading. In Section 4, we propose an alternative, threshold-based strategy in the case that the random variables follow a Brownian motion. In Section 5, we implement the strategy specified in Section 4 and identify empirical competitive ratios generated by the strategy. Lastly, in Section 6, we summarize our findings and propose potential areas for improving competitive ratios with respect to trading.

2 Background

We review basic options theory in Section 2.1; namely, we provide an overview of options, option variants, and standard pricing methods. Additionally, we provide some brief results from probability in Section 2.2 that will help us in later sections. Lastly, we analyze common variants of the prophet inequality in Section 2.3, how they differ from the original prophet inequality, and the known bounds for each variant. The results of Section 2.2 are most pertinent to developing our stopping rule for option exercise, though Sections 2.1 and 2.3 are tangentially related and provide perspective on more realistic market models that could be used in the future and on current prophet inequality literature, respectively.
2.1 Derivatives and Options

Formally, a financial derivative is a contract whose value is dependent upon (or derived from) a specified asset or set of assets. These asset(s) are collectively known as the underlying. As of June 2019, over-the-counter derivatives trading accounted for almost $640 trillion of notional (underlying) value [1]. For reference, the United States GDP in 2019 was $21.44 trillion. Despite their modern popularity, derivatives have existed for millennia; in Ancient Mesopotamia, the Code of Hammurabi described forward contracts allowing the sale of goods at a set price at a later date. Thus, as early as 1750 BCE, people have depended on derivatives to secure prices for assets. Without derivatives, market participants are more exposed to underlying price fluctuation, threatening financial solvency in the event that prices climb too high (e.g. manufacturers cannot afford raw materials) or fall too low (e.g. suppliers cannot profitably generate raw materials). As such, derivatives have played a central role in the development of modern society and continue to serve as a key component of business infrastructure.

Currently, some of the most widely traded derivatives are forwards, futures, and options. Forward contracts are an agreement between two parties, the buyer and the seller, to purchase a product at a given price on a later date. Essentially, the buyer and seller agree in the present on the terms of a future transaction. Futures contracts are effectively the same as forward contracts, working to lock in a price for both buyer and seller, though they differ mechanically. Options, however, differ from both forwards and futures by offering the holder of an option the right to use or exercise their contract. Starting with the most basic form of an option, we say that a call is the right to buy a security \( S \) at a set price \( K \), known as the strike, at a later date \( t \), known as expiry. The sister contract, the put, is identical except that it holds the right to sell the security. Given that the holder of a call or put will only exercise if the forward price at expiry \( S_t \) is above or below the strike, respectively, the payoff can be written as \( \max\{S_t - K, 0\} \) for calls and \( \max\{K - S_t, 0\} \) for puts. The payoff for each contract is illustrated in Figures 1 and 2. When an option has positive payoff, we say it is in the money. Conversely, an option with negative payoff is called out of the money. Lastly, an option with zero payoff (i.e. \( S_t = K \)) is at the money.

2.1.1 Option Styles

The option variant described above is known as a European option. More complex variants exist, differing with respect to exercise or payoff. One of the most popular variants, the American option, allows option holders to exercise any time before expiry. Other “exotic” options include Asian options, which depend on an average of underlying prices, barrier options, which void upon the underlying reaching a certain price, and lookback options, which depend on the maximum or minimum price achieved by the underlying during the lifetime of the option. We focus on the Bermuda option, which allows exercise at a predetermined set of times leading up to expiry. For example, the holder of a Bermudan option may be able to exercise their option on the first day of every month. Thus, the Bermudan option lies somewhere between a European and an American option, just as Bermuda lies somewhere between Europe and America. Note that this alone is its namesake; Bermuda options are not limited to Bermuda and are traded widely in global markets.
Figure 1: Payoff diagram for a call with strike $K$.

Figure 2: Payoff diagram for a put with strike $K$. 
2.1.2 Option Pricing

Though options and their variants have existed for quite a while, the problem of appropriately pricing options remains an active area of research. In 1973, Black and Scholes [2] published the famous Black-Scholes model for pricing European options, which provided great insight into options pricing. Given only the volatility of the underlying, Black-Scholes implied that there existed a single unique price for the option. Fundamentally, however, Black-Scholes made significant assumptions that do not accurately reflect real markets. Specifically, Black-Scholes assumes that the underlying follows a geometric Brownian motion with known constant drift and volatility. This, in turn, implies that the prices and thus the returns of the underlying follow a log-normal distribution, effectively blinding Black-Scholes to latent skewness or kurtosis in the underlying return distribution. Indeed, many securities exhibit skewness and kurtosis in their return distributions. Particularly, Black-Scholes diverges for far out of the money options, as would be expected from ignoring skewness and kurtosis. Additionally, Black-Scholes makes it difficult to adjust for changing drift and volatility, as well as alternative exercise or payoff styles.

While Black-Scholes itself is known to be flawed, it has served as a foundation for many common pricing models used today. Many such models retain the basic assumption that the underlying can be modeled as a stochastic process, but make modifications either with respect to the type of process or with payoffs at specific steps in the process. Commonly, modern models employ binomial tree or Monte Carlo (simulation) methods.

2.1.3 Binomial Options Pricing Model

The binomial options pricing model (BOPM), first formalized by Cox, Ross, and Rubinstein [6], reduces the price of the underlying to a random walk. Given a desired tree depth and initial underlying value of $S$, we specify multipliers $u$ and $d$ corresponding to upward and downward moves, respectively, such that $d = 1/u$. Then, under no-arbitrage, we can solve for the probability $p$ of an upward move as

\[ psu + (1-p)sd = S \]
\[ pu + d - pd = 1 \]
\[ p = \frac{1-d}{u-d}. \]

We can then generate a binomial tree where each node $S_{i,j}$, beginning with root $S_{0,0}$, has two children $S_{i+1,j}$ and $S_{i+1,j+1}$ such that $S_{i+1,j} = dS_{i,j}$ and $S_{i+1,j+1} = uS_{i,j}$. The edges from parent to child are weighted with $p$ and $1-p$ accordingly; an example tree is provided in Figure 3. To value the option, we propagate payoffs from terminal nodes to the root of the tree. At terminal nodes, this is simple as the payoff at that node is itself the maximum expected value. At interior nodes, we take the expected value of its children (if exercise is not available) or the maximum of the expected value of its children and its immediate exercise value (if exercise is available). As we increase the depth of the tree, the value of a European option according to BOPM converges to its value under Black-Scholes [6]. However, the BOPM also allows us to model more complex underlying movements (e.g. dividend payments, which decrease the value at a level by a known amount) or exercise styles (e.g. American options, which allow exercise at any time) than Black-Scholes, giving it a slight advantage.
2.1.4 Longstaff-Schwartz Model

The Longstaff-Schwartz Model (LSM) introduced by Longstaff and Schwartz [12] simulates a large number of potential underlying paths and, aggregating over all paths, computes a price for the option. Since LSM does not require fixed price movements, LSM may be better able to mimic the underlying on any given path. Similar to the BOPM, LSM works from future time steps to the present. At each time step and for each path, LSM computes values for the option by regressing against the exercise payoff for all in the money paths. These values are compared against the immediate exercise values; the greater of the two is taken as the value at that time step and path. Then, when computation has reached the beginning of the paths, LSM takes the average of the first exercise along each path as the value of the option.

2.2 Probability

We will find the following results useful.

**Theorem 2.1** (Inverse Mills Ratio). Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then for $\alpha > 0$,

\[
\mathbb{E}[X \mid X > \alpha] = \mu + \sigma \frac{\phi\left(\frac{\alpha - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\alpha - \mu}{\sigma}\right)},
\]

\[
\mathbb{E}[X \mid X < \alpha] = \mu - \sigma \frac{\phi\left(\frac{\alpha - \mu}{\sigma}\right)}{\Phi\left(\frac{\alpha - \mu}{\sigma}\right)},
\]

where $\phi$ and $\Phi$ denote the standard normal probability density function and the standard normal cumulative distribution function, respectively.

**Theorem 2.2.** Let $X_1, \ldots, X_n$ be i.i.d. and let $S_k = \sum_{i=1}^{k} X_i$. Then, letting $S_i^+ = \max S_i, 0$,

\[
\mathbb{E}[\max_i S_i^+] = \sum_{i=1}^{n} \frac{1}{i} \cdot \mathbb{E}[S_i^+].
\]
This is shown by Spitzer [13] as Corollary 1 to Theorem 4.1.

**Theorem 2.3.** Let $B_t$ be a Brownian motion with variance $\sigma^2$ and drift parameter $\mu$. Assuming $B_t$ starts at some $B_0 \in \mathbb{R}$, for any $n \in \mathbb{N}$, the vector $(B_1, \ldots, B_n)$ is multivariate Gaussian with mean $\mu$ and covariance $\Sigma$ such that $\mu_i = B_0 + i\mu$ and $\Sigma_{i,j} = \sigma^2 \min\{i, j\}$.

**Remark 2.1.** The probability density function of a multivariate Gaussian variable in $\mathbb{R}^n$ drawn from $\mathcal{N}(\mu, \Sigma)$ can be written

$$f(x) = \frac{\exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)}{\sqrt{(2\pi)^n \det \Sigma}}.$$ 

### 2.3 Prophet Inequality and Variants

The original prophet inequality discussed previously was a seminal result that prompted the study and analysis of pathological competitive ratios and their upper bounds in problems throughout optimal stopping theory. Recall that, given a sequence of known distributions $F_1, \ldots, F_n$ and corresponding samples $X_1, \ldots, X_n$ from those distributions, the prophet inequality describes how we can bound the winnings of a gambler who must accept or reject a sample upon receiving it (without knowledge of future samples) to that of a prophet who knows the realizations of all random variables. The gambler has some strategy or stopping rule to maximize the expected value of the selected sample. Krengel and Sucheston [10, 11] identified a $1/2$-competitive strategy using a simple threshold rule (i.e. the gambler accepts a sample only if it exceeds a chosen threshold). The pathological competitive ratio of $1/2$ serves as an important baseline with which we can evaluate the performance of certain prophet inequality variants. Since prophet variants are often special cases of the original inequality, we know trivially that the gambler should be able to achieve a pathological competitive ratio of at least $1/2$ without modifying their strategy to take advantage of the new conditions. Though many variants of the prophet inequality are studied, we highlight recent findings for I.I.D., order selection, prophet secretary, and correlated prophets in the remainder of this section.

#### 2.3.1 I.I.D.

One notable variant of the prophet inequality restricts the nonnegative random variables $X_1, \ldots, X_n$ to be i.i.d., hence the name. In other words, the gambler independently samples a single distribution (as opposed to several, distinct distributions) finitely many times. As any problem of this form is also an instance of the original prophet inequality, we should again expect to find a pathological competitive ratio of at least $1/2$. Hill and Kertz [9] proved an upper bound on the pathological competitive ratio of $1/\beta \approx 0.7451$ where $\beta$ is the solution to

$$\int_0^{1/\beta} \frac{1}{y(1 - \ln y) + \beta - 1} \, dy = 1.$$ 

Recently, Correa et al. [4] identified a $1/\beta$-competitive strategy, confirming that the upper bound was tight.

#### 2.3.2 Order Selection

In this variant of the prophet inequality, the gambler can select the order in which to sample the variables. This relaxation, naturally, should allow the gambler to expect at least as great
a proportion of the prophet’s winnings as in the original problem; that is, we should be able to establish a pathological competitive ratio of at least $1/2$. Chawla et al. [3] first improved the pathological competitive ratio to $1 - 1/e$, which has recently been further improved to $1 - 1/e + 1/30 \approx 0.6655$ by Correa, Saona, and Ziliotto [5]. As a sequence of i.i.d. variables offers no advantage in terms of order selection, the upper bound of $1/\beta$ is the best upper bound known on the pathological competitive ratio for the order selection variant [5].

2.3.3 Prophet Secretary

Before discussing the prophet secretary problem, we must discuss the secretary problem. In its canonical form, the secretary problem describes the following challenge. Given that we are looking to hire a secretary, we evaluate candidates in some random order and, after each interview, we must decide immediately whether to send the candidate away (without the option of calling them back) or to hire the candidate, thus stopping us from examining future candidates. Thus, one key difference between the secretary problem and the prophet inequality is that the gambler has knowledge of future distribution in prophet inequality settings.

Combining the random order of evaluation from the secretary problem and knowledge over sample distributions from the original prophet inequality, we arrive at the prophet secretary problem. Whereas the gambler originally sampled distributions in some arbitrary but known order, the gambler must now sample distributions in a random order. Esfandiari et al. [8] first introduced prophet secretary, obtaining a pathological competitive ratio of $1 - 1/e$ with a stopping rule consisting of a series of thresholds. Ehsani et al. [7] showed that the gambler can achieve a pathological competitive ratio of $1 - 1/e$ with only one threshold, conditioned on the fact that ties could be broken arbitrarily. Most recently, Correa et. al. [5] improved the pathological competitive ratio to approximately $1 - 1/e + 1/27 \approx 0.669$. Again, since randomizing the order of the variables has no effect when they are i.i.d., $1/\beta$ remains the lowest known upper bound for pathological competitive ratios in the prophet secretary variant [5].

2.3.4 Correlated Prophets

Correlated prophets stand out from the previously mentioned variants by allowing correlation between the variables. Truong and Wang [15] devised a $1/(1 + \bar{T})$-competitive strategy where $\bar{T}$ represents the expected number of stopping opportunities available to the gambler. We can upper bound competitive ratios for correlated prophet inequalities by noting that the example shown in Section 1 is a special case of (un)correlated distributions. Recall that, letting $X_1 = 1$ and $X_2$ be $1/p$ and 0 with probability $p$ and $1 - p$, respectively, no strategy can expect a competitive ratio of more than $1/2$. Notably, [15] bounds the prophet’s expected value by summing the expected value at each time step, effectively allowing the prophet to stop at each step. We expand on this in the following section.
3 Prophet Application to Trading

In this section, we consider how each of the prophet inequality variants discussed in Section 2.3 could be applied to trading. Specifically, we consider potential trading strategies that benefit from the stopping rules suggested by each variant.

Notably, each of the order selection, i.i.d., and prophet secretary variants of the prophet inequality require independence between the random variables. While each variant offers an improved competitive ratio over the original prophet inequality (which also requires independence), it is difficult to apply such findings to the field of trading. It is rare to find a security without some degree of autocorrelation or correlation to external factors, however arbitrary these may be. For these reasons, we focus more on the findings of Truong and Wang [15], which allow for correlations between both the random variables and with external factors.

As stated previously, [15] obtained a $1/(1 + T)$-competitive strategy for arbitrarily correlated distributions. However, their findings remained bounded above by $1/2$ in the same manner as the original inequality using uncorrelated variables. For shorter-dated options (i.e. fewer opportunities to exercise), the strategy in [15] may be viable as $T$ is small. However, as the number of exercise opportunities increases, $T$ increases as well and we would observe decreasing competitive ratios. However, recall from Section 2.3.4 that [15] allowed the prophet to stop at every opportunity when constructing an upper bound on the prophet’s winnings. By simplifying market movements to Brownian motions (with negative drift), we can forego arbitrary correlation and create a stopping rule that obtains competitive ratios beyond the pathological $1/(1 + T)$ suggested by [15].

4 Prophet Inequality with Brownian Motions

Truong and Wang [15] devised a $1/(1 + T)$-competitive algorithm for prophet inequalities with arbitrarily correlated distributions. However, restricting the sequence of random variables $X_1,\ldots,X_n$ to track a Brownian motion $B_t$ such that $X_i = B_i^+$, we were able to devise an alternative algorithm that yields greater empirical competitive ratios $1/(1 + T)$ than [15].

Exercise 4.1. Consider a Brownian motion $\{B_t\}_{t=1}^T$ with variance $\sigma^2$ and drift $\mu < 0$, and let $B_0 \in \mathbb{R}$. If we stop at $t$, we receive $B_t^+ = \max\{B_t, 0\}$. Define a stopping rule that maximizes the expected payoff.

Solution. We define a stopping rule such that we stop at $t$ if

$$B_t^+ \geq V_t(t + 1)$$

where $V_s(t)$ is the expected value of our strategy at $t$ given information at $s$. We can then define a stopping threshold $a_t$ such that we stop if $B_t \geq a_t$ and continue otherwise. More specifically, $a_t$ is the fixed point of the map

$$B_t \mapsto V_t(t + 1).$$

This can be seen from a simple argument; if $B_t \geq a_t$, then $B_t \geq V_t(t + 1)$ and we stop, otherwise if $B_t < a_t$, then $B_t < V_t(t + 1)$ and we continue. Since payoff is always non-negative, we know $V_s(t) \geq 0$ and thus $a_t \geq 0$ for all $s, t \in \{1, \ldots, T\}$. Furthermore, $a_T = 0$.
since we only accept positive values at the last stopping point. Let
\[ \Gamma_s(t) = (\land_{i=s+1}^{t-1} (B_i < a_i)) \land B_s \]
represent the condition we reject stopping points between \( s \) and \( t \) given \( B_s \). We can then write \( V_s(t) \) as the recurrence relation
\[ V_s(t) = \mathbb{P}(B_t \geq a_t \mid \Gamma_s(t)) \cdot \mathbb{E}[B_t \mid B_t \geq a_t, \Gamma_s(t)] + \mathbb{P}(B_t < a_t \mid \Gamma_s(t)) \cdot V_s(t + 1). \]
As a terminal condition, we write
\[ V_s(T) = \mathbb{E}[B_T^+ \mid B_s] = \mathbb{P}(B_T \geq 0 \mid \Gamma_s(T)) \cdot \mathbb{E}[B_T \mid B_T \geq 0, \Gamma_s(T)]. \]
Note that \( V_s(t) \) is parameterized by \( a_i \) for \( i \in \{t, \ldots, T\} \). Since \( a_i \) is the fixed point of \( B_t \mapsto V_t(t+1) \) which depends only on \( a_i \) for \( i > t \), we can iterate fixed points backwards from \( a_T \) to \( a_1 \) to determine the sequence \( \{a_i\}_{t=1}^T \). That is, given \( V_0(T) \), we can find \( a_{T-1} \), giving us \( V_0(T-1) \), letting us find \( a_{T-2} \), and so forth. It remains to find \( V_0(1) \). We expand
\[ V_0(1) = \mathbb{P}(B_1 \geq a_1 \mid \Gamma_0(1)) \cdot \mathbb{E}[B_1 \mid B_1 \geq a_1, \Gamma_0(1)] + \mathbb{P}(B_1 < a_1 \mid \Gamma_0(1)) \cdot V_0(2), \]
where we let \( \prod_{j=1}^{i-1} \mathbb{P}(B_j < a_j \mid \Gamma_0(j)) = 1 \) if \( i = 1 \). The product simplifies below to
\[ \prod_{j=t}^{i-1} \mathbb{P}(B_j < a_j \mid \Gamma_s(j)) = \mathbb{P}(B_t < a_t \mid \Gamma_s(t)) \cdot \prod_{j=t+1}^{i-1} \mathbb{P}(B_j < a_j \mid \Gamma_s(j)) \]
\[ = \mathbb{P}(B_t < a_t, B_{t+1} < a_{t+1} \mid \Gamma_s(t)) \cdot \prod_{j=t+2}^{i-1} \mathbb{P}(B_j < a_j \mid \Gamma_s(j)) \]
\[ = \mathbb{P}(B_t < a_t, \ldots, B_{i-1} < a_{i-1} \mid \Gamma_s(t)) \]
for arbitrary \( s, t \) with \( s < t \). We can then further simplify to find
\[ V_0(1) = \sum_{i=1}^{T} \left( \prod_{j=1}^{i-1} \mathbb{P}(B_j < a_j \mid \Gamma_0(j)) \right) \cdot \mathbb{P}(B_i \geq a_i \mid \Gamma_0(i)) \cdot \mathbb{E}[B_i \mid B_i \geq a_i, \Gamma_0(i)]. \]
We can use Remark 2.1 to see $E[B_i \cdot 1_{B_i \geq a, \Gamma_0(i)} \mid B_0]$ is given by
\[
\int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_{i-1}} \int_{a_i}^{\infty} x_i \cdot \frac{\exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)}{\sqrt{(2\pi)^i \det \Sigma}} \, dx.
\]
Furthermore, from Theorem 2.3, we know $x = (x_1, \ldots, x_i)$, $\mu = (B_0 + \mu, \ldots, B_0 + i\mu)$, and $\Sigma$ is the $i \times i$ covariance matrix with $u, v$ element $\Sigma_{u,v} = \sigma^2 \min\{u, v\}$.

More generally, fixing $s < T$ (and again by Remark 2.1 and Theorem 2.3), we see that
\[
V_s(t) = \sum_{i=t}^{T} E[B_i \cdot 1_{B_i \geq a, \Gamma_s(i)} \mid B_s],
\]
$E[B_i \cdot 1_{B_i \geq a, \Gamma_s(i)} \mid B_s]$ is given by
\[
\int_{-\infty}^{a_{s+1}} \cdots \int_{-\infty}^{a_{i-1}} \int_{a_i}^{\infty} x_i \cdot \frac{\exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)}{\sqrt{(2\pi)^i \det \Sigma}} \, dx
\]
where
\[
x = (x_{s+1}, \ldots, x_i),
\mu = (B_s + \mu, \ldots, B_s + (i-s)\mu),
\Sigma_{u,v} = \sigma^2 \min\{u, v\}.
\]

Exercise 4.2. What would the prophet’s payoff be in Exercise 4.1?

Solution. We must determine the expected payoff of the prophet, or
\[
E\left[\max_{1 \leq t \leq T} B_t^+\right].
\]
Logically, $B_0$ is either zero or nonzero. If $B_0 = 0$, this is quite simple. Let $\{X_t\}_{t=1}^T$ be an i.i.d. sequence of normally distributed variables $\mathcal{N}(\mu, \sigma^2)$. By definition, we can write $B_t = \sum_{i=1}^{t} X_i$. Applying Lemma 2.2, we see
\[
E\left[\max_{1 \leq t \leq T} B_t^+\right] = \sum_{t=1}^{T} \frac{1}{t} \cdot E[B_t^+].
\]
Since $B_0 = 0$, we know each $B_t \sim \mathcal{N}(t\mu, t\sigma^2)$ and thus
\[
E[B_t^+] = \mathbb{P}(B_t \geq 0) \cdot E[B_t \mid B_t \geq 0, B_0 = 0],
\]
\[
= \left(1 - \Phi\left(-\frac{\mu\sqrt{t}}{\sigma}\right)\right) \cdot \left(t\mu + \sigma\sqrt{t} \cdot \frac{\phi\left(-\frac{\mu\sqrt{t}}{\sigma}\right)}{1 - \Phi\left(-\frac{\mu\sqrt{t}}{\sigma}\right)}\right),
\]
\[
= \Phi\left(\frac{\mu\sqrt{t}}{\sigma}\right) \cdot t\mu + \phi\left(-\frac{\mu\sqrt{t}}{\sigma}\right) \cdot \sigma\sqrt{t}.
\]
by the inverse Mills ratio.

If $B_0 \neq 0$, the solution is less trivial, but we can find a suitable upper bound for the winnings of the prophet. We write each $B_t = B_0 + B'_t$ where $B'_t = \sum_{i=1}^{t} X_i$. Then, in the case that $B_0 < 0$, we have

$$
\mathbb{E}\left[ \max_{1 \leq t \leq T} B'^+_t \right] = \mathbb{E}\left[ \max_{1 \leq t \leq T} (B_0 + B'_t)^+ \right]
\leq \mathbb{E}\left[ \max_{1 \leq t \leq T} B'^+_t \right]
= \sum_{t=1}^{T} \frac{1}{t} \cdot \mathbb{E}[B'^+_t]
$$

where we can again calculate $B'^+_t$ as we did when $B_0 = 0$. Thus, the above is an upper bound when $B_0 < 0$.

In the case that $B_0 > 0$, we have

$$
\mathbb{E}\left[ \max_{1 \leq t \leq T} B'^+_t \right] = \mathbb{E}\left[ \max_{1 \leq t \leq T} (B'_t + B_0)^+ \right]
\leq B_0 + \mathbb{E}\left[ \max_{1 \leq t \leq T} B'^+_t \right]
= B_0 + \sum_{t=1}^{T} \frac{1}{t} \cdot \mathbb{E}[B'^+_t].
$$

Again, we have an upper bound for the prophet. Noting that $B_0^+ = 0$ if $B_0 \leq 0$ and $B_0^+ = B_0$ if $B_0 > 0$, we can combine the above expressions to obtain a single upper bound for the prophet

$$
\mathbb{E}\left[ \max_{1 \leq t \leq T} B'^+_t \right] \leq B_0^+ + \sum_{t=1}^{T} \frac{1}{t} \cdot \mathbb{E}[B'^+_t],
$$

with equality if $B_0 = 0$. ◀

5 Algorithm and Implementation

While we were able to identify an expression for the expected value of both our strategy and the prophet in Section 4, the expected value of our strategy is notably difficult and expensive to compute when the number of time steps $T$ grows large as the required number of integrals to evaluate grows $O(T^2)$. In practice, we would like to compute at least the expected value of our strategy as quickly as possible. We resort to numerical methods to produce highly accurate estimates. The remainder of this section is organized as follows: in Section 5.1, we discuss the methodology with which we produce estimates, and in Section 5.2, we present preliminary results.
5.1 Methodology

Since we established an exact formula for the prophet’s payoff when the initial value \( B_0 = 0 \) and an upper bound for the prophet’s payoff when \( B_0 \neq 0 \) in Exercise 4.2, it remains to compute the payoff of our strategy in Exercise 4.1. Instead of computing the continuous integral given at the end of Exercise 4.1, we discretize the possible values of the Brownian process (and the corresponding moves) and directly compute the expected value of our strategy. As the granularity of the discretization increases, the computed value should approach the true value of the integral.

Let \( \{B_t\}_{t=1}^T \) be a Brownian motion described by some normal random variable \( X \sim \mathcal{N}(\mu, \sigma^2) \). We can then define a discrete random variable \( Y_c \) with support \( \mathbb{Z}/c = \{z/c \mid z \in \mathbb{Z}\} \) for \( c \in \mathbb{R}^+ \). The corresponding mass function is given below as

\[
P(Y_c = x) = P\left( x - \frac{1}{2c} \leq X < x + \frac{1}{2c} \right).
\]

for \( x \in \mathbb{Z}/c \). Clearly, \( Y_c \) converges weakly to \( X \) as \( c \to \infty \). Then, using \( Y_c \), let \( \{W_t\}_{t=1}^T \) be the random walk where

\[W_{t+1} = W_t + Y_c \]

and \( W_0 = \arg \min_{z \in \mathbb{Z}/c} |B_0 - z| \) is the element in \( \mathbb{Z}/c \) closest to \( B_0 \). Since \( Y_c \to X \) as \( c \to \infty \), we should have \( W_t \to B_t \) as well. Then calculating the expected payoff of our strategy on \( W_t \) should converge to that of our strategy on \( B_t \) as we let \( c \) grow arbitrarily large. Let \( U_t(z/c) \) denote the expected value of our strategy at \( t \) given \( W_t = z/c \). By definition, we must have

\[
U_t\left(\frac{z}{c}\right) = \max\left\{\frac{z}{c}, \mathbb{E}\left[U_{t+1}\left(\frac{z}{c} + Y_c\right)\right]\right\},
\]

but

\[
\mathbb{E}\left[U_{t+1}\left(\frac{z}{c} + Y_c\right)\right] = \sum_{x \in \mathbb{Z}/c} U_{t+1}\left(\frac{z}{c} + x\right) \cdot P(Y_c = x),
\]

so we can compute \( U_t(z/c) \) recursively given \( U_{t+1}(\mathbb{Z}/c) \). Additionally,

\[
U_T\left(\frac{z}{c}\right) = \max\left\{\frac{z}{c}, 0\right\} = \left(\frac{z}{c}\right)^+
\]

since \( T \) is the final stopping time. Letting \( P(x) = \mathbb{P}(Y_c = -x) \), we can rewrite the above expression as

\[
\mathbb{E}\left[U_{t+1}\left(\frac{z}{c} + Y_c\right)\right] = \sum_{x \in \mathbb{Z}/c} U_{t+1}\left(\frac{z}{c} + x\right) \cdot \mathbb{P}(Y_c = x),
\]

\[
= \sum_{x \in \mathbb{Z}/c} U_{t+1}\left(\frac{z}{c} + x\right) \cdot P(-x),
\]

\[
= (U_{t+1} \ast P)\left(\frac{z}{c}\right).
\]

We can then run the algorithm recursively on \( t \) to find \( U_{T-1}(\mathbb{Z}/c) \), \( U_{T-2}(\mathbb{Z}/c) \), and so forth until we compute \( U_1(\mathbb{Z}/c) \).
Instead of an integral, we must now compute an infinite sum (which is not much better). However, we can bound $W_t$ to the region $S_n = [-n, n]$ such that $\mathbb{P}(W_t \not\in S_n)$ and, accordingly, $\mathbb{P}(Y_c \not\in S_n)$ are vanishingly small (e.g. cannot be represented within floating point precision) to convert this to a finite sum with reasonable accuracy. We provide a general outline in Algorithm 1.

**Algorithm 1** Numerically compute the expected value of our strategy.

```
function Numeric(T: the number of time steps to compute)
    ∀z ∈ S_n, U(\hat{z}) ← (\hat{z})^+
    while T > 0 do
        ∀z ∈ S_n, U(\hat{z}) ← max\{U(\hat{z}), (U * P)(\hat{z})\}
        T ← T - 1
    end while
    return U
end function
```

### 5.2 Results

For $B_0 = 0$, $X \sim \mathcal{N}(-1, 1)$, and varying maximum time steps $T$, we observed the competitive ratios shown in Figure 4. As $T \to \infty$, the competitive ratio exhibits asymptotic behavior.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Competitive Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.88709</td>
</tr>
<tr>
<td>3</td>
<td>0.84838</td>
</tr>
<tr>
<td>4</td>
<td>0.83190</td>
</tr>
<tr>
<td>5</td>
<td>0.82419</td>
</tr>
<tr>
<td>10</td>
<td>0.81667</td>
</tr>
<tr>
<td>100</td>
<td>0.81633</td>
</tr>
<tr>
<td>1000</td>
<td>0.81633</td>
</tr>
</tbody>
</table>

Figure 4: Empirical competitive ratios from $\mathcal{N}(-1, 1)$.

Varying the drift $\mu$ and variance $\sigma^2$ and fixing $T = 100$, we observe the competitive ratios shown in Figure 5.

The competitive ratio exhibits asymptotic behavior as $\mu/\sigma \to 0$. Interestingly, the competitive ratios are non-monotonic; there exists a (local) minimum at some point between $\mu/\sigma = -0.2$ and $\mu/\sigma = -0.01$. Since the competitive ratio depends only on the ratio $\mu/\sigma$, we expect that for large $\sigma$ or small $\mu$, the ratios converge at $\mu = 0$ for any $\sigma$. This was verified for $\mu = 0$ and varying $\sigma$, with a competitive ratio of 0.53797 in all instances.

### 6 Conclusion

While many variants of the prophet inequality have been studied extensively as seen in Section 2.3, little work has addressed prophet inequalities with correlated distributions. Truong
## Figure 5: Empirical competitive ratios with varying $\mu, \sigma^2$.

and Wang [15] addressed inequalities that allow arbitrary correlation between samples, establishing a pathological competitive ratio of $1/(1 + \bar{T})$.

In Section 4, we explored prophet inequalities in which the sequence of samples $B_1, \ldots, B_T$ followed a modified Brownian motion with payoff $B^+_t$ at each time $i$. If the process has initial value $B_0 \in \mathbb{R}$, drift $\mu < 0$, and variance $\sigma^2$, we know the prophet to have an expected payoff of at most

$$B^+_0 + \sum_{t=1}^T \frac{1}{t} \cdot \mathbb{E}[B^+_t],$$

where $B^+_t$ is the same process as $B_t$ but with initial value $B'_0 = 0$. Then if we follow the strategy of stopping when future expected payoff is below the current (immediate) payoff and rejecting otherwise, we obtain an expected payoff of

$$\sum_{t=1}^T \mathbb{P}(B_t \geq a_t \wedge (\wedge_{i=1}^{t-1} B_i < a_i) \mid B_0) \cdot \mathbb{E}[B_t \mid B_t \geq a_t \wedge (\wedge_{i=1}^{t-1} B_i < a_i) \wedge B_0]$$

or, more tersely,

$$\sum_{t=1}^T \mathbb{E} \left[ B_t \cdot 1_{B_t \geq a_t} \cdot \prod_{i=1}^{t-1} 1_{B_i < a_i} \mid B_0 \right],$$

where $a_t$ is the minimum value at which we would prefer stopping at $B^+_t$ over continuing to $B^+_{t+1}$. Intuitively, we can think of this as the sum over each time of the expected payoff of the current time $B^+_t$ given that it exceeds $a_t$ and that every previous sample $B_i$ did not exceed its respective $a_i$, weighted by the corresponding probability.

In Section 5.2, we identified empirical competitive ratios using numerical analysis via Algorithm 1 for varying Brownian motions with negative drift that the pathological $1/(1 + \bar{T})$ competitive ratio shown in [15]. Notably, we can use Algorithm 1 to evaluate the expected value of our strategy for arbitrary movement distributions, including distributions that change over time. By modifying $P$ at each time step, we can represent an extremely wide range of processes (e.g. geometric Brownian motions, a series of random processes, etc.), though

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>Competitive Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$1/10$</td>
<td>0.99999</td>
</tr>
<tr>
<td>-1</td>
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</tr>
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<td>10000</td>
<td>0.53770</td>
</tr>
<tr>
<td>-1</td>
<td>100000</td>
<td>0.53794</td>
</tr>
</tbody>
</table>
finding an expression for the strategy’s expected value as in Section 4 may be more difficult depending on the process used.

While we were able to establish expressions for competitive ratios for Brownian motions with negative drift, we did not identify the pathological Brownian motion nor the corresponding competitive ratio. Additionally, it would be interesting to see how more complex processes affect both optimal strategy and the corresponding competitive ratios. Using more complex processes may be necessary to bridge the gap between theory and application, particularly in the context of trading. In the financial literature, for example, it is commonly assumed that securities follow geometric Brownian motions. Empirically, the log-returns of many securities are nonnormal, suggesting that more esoteric distributions may be necessary; for instance, Taleb [14] suggests that security returns follow a power law distribution. Also, it is likely that the return distribution of a security is highly sensitive to events. For example, positive news regarding a company’s revenues for the quarter are likely to cause a sudden, one-time jump in the company’s share price. Such a change is likely better accounted for separately from a probabilistic process. Much work is still necessary to identify processes that best model price movements of financial securities. Then, assuming that an ideal process is found, it may be difficult to identify parameters that best fit the security in question. Depending on how widely a security’s true return distribution differs from theoretical distributions, there may be more edge in obtaining a more accurate estimate of the true distribution than more optimally trading an less accurate estimate. Optimal trading strategies are thus dependent on both accurate modeling of underlying market processes and prophet-like results.
References


